MTH 264 Introduction to Matrix Algebra - Summer 2023.

LN3B. Determinants.

These lecture notes are mostly lifted from the text **Matrix and Power Series, Lee and Scarborough, custom 5th edition**. This document highlights parts of the text that are used in the lecture sessions.

We start with an algebraic definition for determinants.

Definition 3B.1. Determinants for $n = 2$ and $n = 3$

The **determinant**, $\det(\mathbf{A})$, of a square matrix $\mathbf{A}$ is some scalar defined by certain properties (not covered in this class). We can view the determinant as a function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ given recursively by the dimension n starting at $n = 2$. In the $n = 2$ case, the determinant is given by

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

In the $n = 3$ case, the determinant can be calculated using a Laplace expansion along a row or a column. For this course, we will usually apply the Laplace expansion along the first row or on the first column. That is,

$$\text{Expansion along } \mathsf{row}_1(\mathbf{A}) : \quad \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = (a_1) \det \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} + (-b_1) \det \begin{pmatrix} a_2 & c_2 \\ a_3 & c_3 \end{pmatrix} + (c_1) \det \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$$

$$\text{Expansion along } \mathsf{col}_1(\mathbf{A}) : \quad \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = (a_1) \det \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} + (-a_2) \det \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + (a_3) \det \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix}$$

Definition 3B.2. Laplace Expansion Higher-Order Determinants.

The Laplace expansion extends to the $n \geq 4$ case.

Let $\mathbf{B} \in \mathbb{R}^{n\times n}$. Denote $\mathbf{B}[i, j] \in \mathbb{R}^{(n-1)\times(n-1)}$ as the matrix generated by removing the i^{th} row and the j^{th} column of $\mathbf{B}$. There are two ways to apply a Laplace expansion:

$$\text{For fixed } i \text{, expand along } \mathsf{row}_i(\mathbf{B}) : \quad \det(\mathbf{B}) = \sum_{j=1}^{n} (-1)^{i+j} (\mathbf{B})_{i,j} \det(\mathbf{B}[i, j])$$

$$\text{For fixed } j \text{, expand along } \mathsf{col}_j(\mathbf{B}) : \quad \det(\mathbf{B}) = \sum_{i=1}^{n} (-1)^{i+j} (\mathbf{B})_{i,j} \det(\mathbf{B}[i, j])$$

Note: Other references call $\det \mathbf{B}[i, j]$ as a minor of $\mathbf{B}$.
